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Vibration modes of $3n$-gaskets and other fractals

N Bajorin, T Chen, A Dagan, C Emmons, M Hussein, M Khalil, P Mody, B Steinhurst and A Teplyaev

Department of Mathematics, University of Connecticut, Storrs CT 06269, USA

E-mail: teplyaev@math.uconn.edu

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Abstract

We rigorously study eigenvalues and eigenfunctions (vibration modes) on the class of self-similar symmetric finitely ramified fractals, which include the Sierpinski gasket and other $3n$-gaskets. We consider the classical Laplacian on fractals which generalizes the usual one-dimensional second derivative, is the generator of the self-similar diffusion process, and has possible applications as the quantum Hamiltonian. We develop a theoretical matrix analysis, including analysis of singularities, which allows us to compute eigenvalues, eigenfunctions and their multiplicities exactly. We support our theoretical analysis by symbolic and numerical computations. Our analysis, in particular, allows the computation of the spectral zeta function on fractals and the limiting distribution of eigenvalues (i.e., integrated density of states). We consider such examples as the level-3 Sierpinski gasket, a fractal 3-tree, and the diamond fractal.

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1. Introduction

There is a large body of physics and mathematics literature devoted to analysis on fractals. A small sample of it, containing many more references, is [3, 8, 18, 22, 53] and [1, 30–34, 42, 43, 52, 54–57, 59, 61, 62]. For example, tools for the numerical analysis of the Sierpiński gasket were developed in [12, 24], and fractal antennae were considered in [20, 29, 46, 48]. One of the most recent papers where the random walks on the Sierpiński gasket play a role is [11]. In most of these works fractals provide examples of irregular or scale-invariant media.

In this paper we rigorously study eigenvalues and eigenfunctions (vibration modes) on the class of self-similar fully symmetric finitely ramified fractals. Such studies originated in [49, 50], where it was observed that on the Sierpiński lattice there are highly localized

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eigenfunctions corresponding to eigenvalues of very high multiplicity. Later the spectrum of the Laplacian on the Sierpiński gasket was studied in detail in [21]. The main purpose of our paper is to develop a theoretical matrix analysis, including analysis of singularities, which allows the exact computation of eigenvalues, eigenfunctions and their multiplicities for a large class of complex fractals.

We consider the classical Laplacian on fractals, which generalizes the usual one-dimensional second derivative, is the generator of the self-similar diffusion process (see [5, 6]), and has possible applications as the quantum Hamiltonian. The latter is especially relevant because it was the original motivation of [49, 50], and because it is shown in [59] that the fractals are natural limits of quantum graphs (see [2, 10, 18, 19, 38–40, and references therein]). Our analysis, in particular, allows the computation of the spectral zeta function on fractals by the method of [13, 58], which can potentially allow the use of zeta regularization techniques and their applications (see [17]). We also compute the limiting distribution of eigenvalues (i.e., integrated density of states), which is a pure point measure (except the case of the usual one-dimensional interval, which is amenable to classical analysis). This support has a representation

$$\mathrm{supp}(\kappa) = \mathcal{J}_R \bigcup \mathcal{D},$$

where $\mathcal{J}_R$ is the Julia set of a rational function, which we compute, and $\mathcal{D}$ is a possibly empty set of isolated points (if $\mathcal{D}$ is infinite, it accumulates to $\mathcal{J}_R$). Also, our analysis allows the computation of eigenvalues and eigenfunctions by a highly accurate hierarchical iterative procedure, which does not involve large matrix calculations[1]. We concentrate on vibrations with no constraints or boundary conditions, partially because this is natural if the Laplacian is interpreted as the generator of the diffusion process or a quantum Hamiltonian, and partially because the computation for the Dirichlet Laplacian (with zero boundary conditions) follows exactly the same way and produces similar results, as we demonstrate in the case of level-3 Sierpiński gasket (theorem 5.2 and table 2 in section 5).

Our study is closely related to the analysis of self-similar graphs [35–37, 44, 45, 51, and references therein], self-similar groups [7, 25–27, 47, 60, and references therein] and the relation between electrical circuits and Markov chains [9, 15, 16, and references therein].

This paper is organized as follows. In section 2 we give the definition of the finitely ramified fractals with full symmetry, on which the graphs which we consider are based. In section 3 we introduce spectral self-similarity, Schur complement and a Dirichlet-to-Neumann map, and show how the resolvent of the Laplacian can be computed by an iterative procedure. In section 4 we analyze the singularities of our map and obtain general formulae for eigenvalues and their multiplicities. We also obtain formulae for corresponding eigenprojectors. In the subsequent sections we use our general method to analyze the following examples: the level-3 Sierpiński gasket (section 5), a fractal 3-tree (section 6) and the diamond fractal (section 7).

2. Finitely ramified fractals with full symmetry

A compact connected metric space F is called a finitely ramified self-similar set if there are injective contraction maps

$$\psi_1, \ldots, \psi_m : F \to F$$

such that

$$F = \bigcup_{i=1}^{m} \psi_i(F)$$

[1] see http://www.math.uconn.edu/~teplyaev/fractals/.

and for any n and for any two distinct words $w, w' \in W_n = \{1, \ldots, m\}^n$ we have

$$F_w \cap F_{w'} = V_w \cap V_{w'},$$

where $F_w = \psi_w(F)$ and $V_w = \psi_w(V_0)$. It is assumed that V_0 is a finite set of at least two points, which often is called the boundary of F. Here for a finite word $w = w_1 \cdots w_n \in W_n$ we denote

$$\psi_w = \psi_{w_1} \circ \cdots \circ \psi_{w_n}.$$

We define

$$V_n = \bigcup_{i=1}^{m} \psi_i(V_{n-1}) = \bigcup_{w \in W_n} V_w$$

and call this set the vertices of level or depth n.

There is a natural infinite self-similar sequence of 'fractal' finite graphs G_n with vertex set V_n defined as follows. For each $n \geqslant 0$ and $w \in W_n$ we define G_w as a complete graph with vertices V_w. Then, by definition,

$$G_n = \bigcup_{w \in W_n} G_w.$$

Note that G_n has no loops, but is allowed to have multiple edges, depending on the structure of the fractal F. The degree of a vertex x in graph G_n is denoted by $\deg_n(x)$. Note that there need not be any uniform bound on the degree of vertices, see for example the Diamond fractal in section 7.

The main object of our study are eigenvalues and eigenfunctions on the probabilistic graph Laplacians Δ_n on G_n, which are defined by

$$\Delta_n f(x) = f(x) - \frac{1}{\deg_n(x)} \sum_{(x,y) \in E(G_n)} f(y),$$

where $E(G_n)$ denotes the set of edges of the graph G_n. For convenience we denote the matrix of Δ_n by M_n in the standard basis of functions on V_n.

Our main geometric assumption is that for any permutation $\sigma : V_0 \to V_0$ there is an isometry $g_\sigma : F \to F$ that maps any $x \in V_0$ into $\sigma(x)$ and preserves the self-similar structure of F. This means that there is a map $\widetilde{g}_\sigma : W_1 \to W_1$ such that

$$\psi_i \circ g_\sigma = g_\sigma \circ \psi_{\widetilde{g}_\sigma(i)}$$

for all $i \in W_1$. The group of isometries g_σ is denoted by $\mathcal{G}$.

It is well known that the eigenvalues and eigenfunctions of Δ_n describe vibration modes of so-called cable systems modeled on the graph G_n, however a free floating crystal lattice would provide the same intuition. They also can be considered as discrete approximations to eigenvalues and eigenfunctions of a continuous self-similar Laplacian Δ_μ on F. This continuous self-adjoint Laplacian is the generator of a self-similar diffusion process on F which can be defined in the standard way in terms of a self-similar resistance (Dirichlet) form on F, that is for any f in a suitably defined domain $\operatorname{Dom} \Delta_\mu$ of the Neumann Laplacian we have

$$\mathcal{E}(f, f) = \int_F f \Delta_\mu f \, \mathrm{d}\mu,$$

where μ is the standard suitably normalized self-similar (Hausdorff, Bernoulli) measure on F.

A $\mathcal{G}$-invariant resistance form $\mathcal{E}$ on F is self-similar with energy renormalization factor ρ if for any $f \in \mathrm{Dom}(\mathcal{E})$ we have

$$\mathcal{E}(f, f) = \rho \sum_{i=1}^{m} \mathcal{E}(f_i, f_i).$$

Here we use the notation $f_w = f \circ \psi_w$ for any $w \in W_n$. Such resistance forms in the case of p.c.f. fractals were studied in detail in [31]. The finitely ramified case can be studied in a similar way because of the general results in [32]. In particular, existence and uniqueness, up to a scalar multiplier, of the local regular self-similar $\mathcal{G}$-invariant resistance form $\mathcal{E}$ is shown in [59]. Moreover, one can show that

$$\mathcal{E} = \lim_{n\to\infty} \rho^{-n}\mathcal{E}_n$$

where the usual graph energy is

$$\mathcal{E}_n(f, f) = \sum_{(x,y)\in E(G_n)} (f(x) - f(y))^2$$

and that

$$(\rho m)^{-n}\Delta_n f(x) \xrightarrow[n\to\infty]{} \Delta_\mu f(x)$$

for any function f for which $\Delta_\mu f \in C(F)$ and any $x \in V_* = \cup_{n\geqslant 0} V_n$. In addition, one has a relation

$$\rho m = \frac{\mathrm{d}}{\mathrm{d}z} R(0) > 1,$$

where $R(z)$ is the rational function that appears in the spectral decimation process, and is one of the most important objects in our study.

The standard and almost trivial example of the self-similar energy and Laplacian in a finitely ramified situation is the case of $F = [0, 1]$. In this case we can take $m = 2$ with $\psi_1(x) = \frac{1}{2}x$ and $\psi_2(x) = \frac{1}{2}x + \frac{1}{2}$, the self-similar measure μ is the usual Lebesgue measure, $\Delta_\mu f = -f''$ and

$$\mathcal{E}(f, f) = \int_0^1 (f'(x))^2\,\mathrm{d}x = \int_0^1 -ff''\,\mathrm{d}x = \int_F f\Delta_\mu f\,\mathrm{d}\mu$$

for any $f \in \mathrm{Dom}(\Delta_\mu) = \{f : f' \in L^2[0, 1], f'(0) = f'(1) = 0\}$. Then we of course have $\rho = 2$ and

$$4^n\Delta_n f(x) = \frac{2f(x) - f\left(x - \frac{1}{2^n}\right) - f\left(x + \frac{1}{2^n}\right)}{4^{-n}} \xrightarrow[n\to\infty]{} -f''(x)$$

for any $f \in C^2[0, 1]$. The cases $F = [0, 1]$ with $m = 3$ and $m = 4$ are discussed in [4].

Although in general the fractal F is an abstract metric space, in our examples $F \subset \mathbb{R}^2$ and the metric on F is the restriction of the usual Euclidean metric in $\mathbb{R}^2$. Moreover, the isometries g_σ are restrictions of isometries of $\mathbb{R}^2$ that maps F into itself and preserves the self-similar structure of F. We do not require that contractions ψ_i be similitudes. One can easily construct more involved and higher dimensional examples for which our methods apply.

3. Spectral self-similarity, Schur complement and Dirichlet-to-Neumann map

If we have a matrix M given in a block form

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{1}$$

then its Schur complement is

$$A - BD^{-1}C. \tag{2}$$

In our work one of the most important objects is the Schur complement of the matrix $M - z$ which is defined by

$$S(z) = A - z - B(D - z)^{-1}C. \tag{3}$$

Note that we use a convention that $M - z$ denotes $M - zI$ where I is the identity matrix of the same size as M. Similarly, $A - z$ and $D - z$ denote the matrices A and D minus z times the identity matrix of the appropriate size.

Our interest in $S(z)$ can be explained as follows. As the initial step in our calculations, we would like to relate the eigenvalues and eigenvectors of the larger Laplacian matrix $M = M_1$ and the eigenvalues and eigenvectors of a smaller Laplacian matrix M_0. In our setup, the blocks A and D in (1) correspond to outer (boundary) and interior vertices, respectively.

Suppose v is an eigenvector of M which is partitioned into its boundary part v_0 and interior part v_1'. Then the eigenvalue equation

$$Mv = zv$$

can be written as

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} v_0 \\ v_1' \end{bmatrix} = z \begin{bmatrix} v_0 \\ v_1' \end{bmatrix} \tag{4}$$

or as two equations

$$\begin{aligned} Av_0 + Bv_1' &= zv_0 \\ Cv_0 + Dv_1' &= zv_1'. \end{aligned} \tag{5}$$

From the second equation we obtain $v_1' = -(D - z)^{-1}Cv_0$, provided $z \notin \sigma(D)$, which implies

$$S(z)v_0 = 0. \tag{6}$$

If v_0 is also an eigenvector of M_0 with an eigenvalue z_0, then we would like to relate (6) with

$$(M_0 - z_0)v_0 = 0. \tag{7}$$

According to [45, 56], we can write $z_0 = R(z)$ if we solve what is our main equation

$$S(z) = \phi(z)(M_0 - R(z)), \tag{8}$$

where $\phi(z)$ and $R(z)$ are scalar (meaning not matrix-valued) rational functions.

Proposition 3.1. *For a given fully symmetric self-similar structure on a finitely ramified fractal F there are unique rational functions $\phi(z)$ and $R(z)$ that solve equation (8).*

Proof. Clearly $S(z)$ is a matrix-valued rational function. By our main symmetry assumption in the previous section, for any z the matrix $S(z)$ is a linear combination of the identity matrix and M_0, which implies the proposition. □

Remark 3.2. From the calculations above one can see that $S(\lambda)$ is the so-called Dirichlet-to-Neumann map for the Laplacian Δ_1.

In our examples M_0 is a matrix that has 1 on the diagonal and $-\frac{1}{N_0-1}$ off the diagonal. Therefore we have that

$$\phi(z) = -(N_0 - 1)S_{1,2}(z)$$

and

$$R(z) = 1 - \frac{S_{1,1}(z)}{\phi(z)}.$$

Here N_0 is the number of boundary vertices, which is the number of points in V_0.

From the calculations above we have the following theorem.

Theorem 3.1. *Suppose that z is not an eigenvalue of D, and not a zero of ϕ. Then z is an eigenvalue of M with an eigenvector v if and only if $R(z)$ is an eigenvalue of M_0 with an eigenvector v_0, and $v = \left[\begin{smallmatrix} v_0 \\ v' \end{smallmatrix}\right]$ where*

$$v' = -(D-z)^{-1}Cv_0.$$

This implies, in particular, that there is an one-to-one map from the eigenspace of M_0 corresponding to $R(z)$ onto the eigenspace of M corresponding to z

$$v_0 \mapsto v = T(z)v_0$$

where

$$T(z) = \begin{bmatrix} I_0 \\ -(D-z)^{-1}C \end{bmatrix}.$$

Naturally, the map $v_0 \mapsto v$ is called the eigenfunction extension map, and $T(z)$ is called the eigenfunction extension matrix.

The theorem above suggests the following definition of the so-called exceptional set

$$E(M_0, M) = \sigma(D) \cup \{z : \phi(z) = 0\}.$$

Once we have computed the functions $R(z)$ and $\phi(z)$ using the smaller matrices M_0 and $M = M_1$, we can compute the spectrum of much larger matrices M_n by induction using the following results.

We use notation

$$M_n = \begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix}$$

for the block decomposition of M_n corresponding to the representation

$$V_n = V_{n-1} \bigcup V_n',$$

where $V_n' = V_n \backslash V_{n-1}$.

Theorem 3.2. *For all $n > 0$ we have a relation*

$$P_{n-1}(M_n - z)^{-1}P_{n-1}^* = \frac{1}{\phi(z)}(M_{n-1} - R(z))^{-1},$$

where P_{n-1} is defined as the restriction operator from V_n to V_{n-1}. We often identify P_{n-1} with the orthogonal projection from $\ell^2(V_n)$ onto the subspace of functions with support in V_{n-1}.

Suppose that $z_n \notin E(M_0, M)$. Then z_n is an eigenvalue of M_n with an eigenvector v_n if and only if

$$z_{n-1} = R(z_n)$$

is an eigenvalue of M_{n-1} with an eigenvector v_{n-1}, and $v_n = \left[\begin{smallmatrix} v_{n-1} \\ v_n' \end{smallmatrix}\right]$ where

$$v_n' = -(D_n - z_n)^{-1}C_n v_{n-1}.$$

In such a situation v_n' is called the continuation of the eigenfunction v_{n-1} from V_{n-1} to $V_n \backslash V_{n-1}$.

One can obtain information about the extension of eigenfunctions and eigenprojectors from V_{n-1} to V_n by the following theorem.

Theorem 3.3. *Let P_{n,z_n} be the eigenprojector of M_n corresponding to an eigenvalue $z_n \notin E(M_0, M)$, and $P_{n-1,z_{n-1}}$ be the eigenprojector of M_{n-1} corresponding to eigenvalue $z_{n-1} = R(z_n)$. Then*

$$P_{n,z_n} = \frac{1}{\phi(z_n)\frac{\mathrm{d}}{\mathrm{d}z}R(z_n)} T_n(z_n) P_{n-1,z_{n-1}}(P_{n-1} - B_n(D_n - z_n)^{-1}P_n'), \tag{9}$$

where

$$T_n(z) = (P_{n-1} - (D_n - z)^{-1}C_n)$$

and P_n' is defined as the restriction operator from V_n to $V_n \backslash V_{n-1}$. We often identify P_n' with the orthogonal projection from $\ell^2(V_n)$ onto the subspace of functions that vanish on V_{n-1}. In this case $P_n' = I_n - P_{n-1}$.

Proof. First we will prove the key formula for the proof of these theorems. This formula is not related to spectral similarity and is a known fact. Essentially, it shows how to find the inverse of a matrix given in a two-by-two block form. To simplify notation we assume that $n = 1$ and $M_1 = M$.

Suppose that matrices $D - x$ and $A - x - B(D - x)^{-1}C$ are invertible. Then $M - x$ is invertible and

$$(M - x)^{-1}(D - x)^{-1} + (P_0 - (D - x)^{-1}C)(A - x - B(D - x)^{-1}C)^{-1}(P_0 - B(D - x)^{-1}). \tag{10}$$

It is enough to prove this formula for $x = 0$, i.e. to prove

$$M^{-1}D^{-1} + (P_0 - D^{-1}C)(A - BD^{-1}C)^{-1}(P_0 - BD^{-1}) \tag{11}$$

provided that D and $A - BD^{-1}C$ are invertible.

We have

$$MD^{-1} = (P_1' + P_0)MD^{-1}P_1' = P_1' + P_0MD^{-1}P_1'$$

and

$$P_0M(P_0 - D^{-1}C) = MP_0 - P_1'MP_0 - P_0MD^{-1}CP_0(A - BD^{-1}C).$$

Thus

$$\begin{aligned} M(D^{-1}P_1' &+ (P_0 - D^{-1}C)(A - BD^{-1}C)^{-1}(P_0 - BD^{-1}P_1')) \\ &= P_1' + P_0MD^{-1}P_1' + P_0(P_0 - BD^{-1}P_1') = P_1' + P_0 = I. \end{aligned}$$

That is what (11) says.

To obtain the proof theorem 3.2, note that (10) implies

$$(M - x)^{-1}(D - x)^{-1}P_1' + (P_0 - (D - x)^{-1}C)(\phi(x)M_0 - \phi_1(x))^{-1}(P_0 - B(D - x)^{-1}P_1'), \tag{12}$$

where $\phi_1(z) = \phi(z)R(z)$. The statements of theorem 3.3 follow if we use the standard spectral representation

$$M = \sum_{z\in\sigma(M)} zP_z$$

and pass to the limit as $x \to z$ in this formula. □

Remark 3.3. For $n = 1$ these theorems are also true for the adjacency matrix graph Laplacian. For $n > 1$ it is important that we consider probabilistic graph Laplacian, or a multiple of it. For instance, [14, 55] and related works usually consider the Laplacian, Δ_n, multiplied by 4.

4. Analysis of the exceptional values

It is not enough to restrict ourself to values of z outside of the exceptional set $E(M_0, M)$. In fact, this set is very interesting because it often contains eigenvalues of high multiplicity, which in turn often correspond to localized eigenfunctions.

We first formulate a proposition that gives the multiplicities of such eigenvalues, and is used extensively to analyze examples in the rest of the paper. Then we prove a theorem which implies the proposition.

We write $\mathrm{mult}_n(z)$ for the multiplicity of z as an eigenvalue of M_n. By definition, $\mathrm{mult}_n(z) = 0$ if z is not an eigenvalue. Notation $\dim_n$ is used for the dimension of $\ell^2(V_n)$ which is the same as the number of points in V_n.

Proposition 4.1.

(i) If $z \notin E(M_0, M)$, then

$$\mathrm{mult}_n(z) = \mathrm{mult}_{n-1}(R(z)), \tag{13}$$

and every corresponding eigenfunction at depth n is an extension of an eigenfunction at depth $n-1$.

(ii) If $z \notin \sigma(D)$, $\phi(z) = 0$ and $R(z)$ has a removable singularity at z, then

$$\mathrm{mult}_n(z) = \dim_{n-1}, \tag{14}$$

and every corresponding eigenfunction at depth n is localized.

(iii) If $z \in \sigma(D)$, both $\phi(z)$ and $\phi_1(z)$ have poles at z, $R(z)$ has a removable singularity at z, and $\frac{\mathrm{d}}{\mathrm{d}z}R(z) \neq 0$, then

$$\mathrm{mult}_n(z) = m^{n-1}\mathrm{mult}_D(z) - \dim_{n-1} + \mathrm{mult}_{n-1}(R(z)), \tag{15}$$

and every corresponding eigenfunction at depth n vanishes on V_{n-1}.

(iv) If $z \in \sigma(D)$, but $\phi(z)$ and $\phi_1(z)$ do not have poles at z, and $\phi(z) \neq 0$, then

$$\mathrm{mult}_n(z) = m^{n-1}\mathrm{mult}_D(z) + \mathrm{mult}_{n-1}(R(z)). \tag{16}$$

In this case $m^{n-1}\mathrm{mult}_D(z)$ linearly independent eigenfunctions are localized, and $\mathrm{mult}_{n-1}(R(z))$ more linearly independent eigenfunctions are extensions of the corresponding eigenfunction at depth $n-1$.

(v) If $z \in \sigma(D)$, but $\phi(z)$ and $\phi_1(z)$ do not have poles at z, and $\phi(z) = 0$, then

$$\mathrm{mult}_n(z) = m^{n-1}\mathrm{mult}_D(z) + \mathrm{mult}_{n-1}(R(z)) + \dim_{n-1} \tag{17}$$

provided $R(z)$ has a removable singularity at z. In this case there are $m^{n-1}\mathrm{mult}_D(z) + \dim_{n-1}$ localized and $\mathrm{mult}_{n-1}(R(z))$ non-localized corresponding eigenfunctions at depth n.

(vi) If $z \in \sigma(D)$, both $\phi(z)$ and $\phi_1(z)$ have poles at z, $R(z)$ has a removable singularity at z, and $\frac{\mathrm{d}}{\mathrm{d}z}R(z) = 0$, then

$$\mathrm{mult}_n(z) = \mathrm{mult}_{n-1}(R(z)), \tag{18}$$

provided there are no corresponding eigenfunctions at depth n that vanish on V_{n-1}. In general we have

$$\mathrm{mult}_n(z) = m^{n-1}\mathrm{mult}_D(z) - \dim_{n-1} + 2\,\mathrm{mult}_{n-1}(R(z)). \tag{19}$$

(vii) If $z \notin \sigma(D)$, $\phi(z) = 0$ and $R(z)$ has a pole z, then $\mathrm{mult}_n(z) = 0$ and z is not an eigenvalue.

(viii) *If $z \in \sigma(D)$, but $\phi(z)$ and $\phi_1(z)$ do not have poles at z, $\phi(z) = 0$, and $R(z)$ has a pole z, then*

$$\mathrm{mult}_n(z) = m^{n-1}\mathrm{mult}_D(z) \tag{20}$$

and every corresponding eigenfunction at depth n vanishes on V_{n-1}.

In the next theorem we establish the relation between eigenprojectors of spectrally similar operators. Namely, we show how one can find the eigenprojector $P_{n,z}$ of M_n corresponding to an eigenvalue z, if the eigenprojector $P_{n-1,R(z)}$ of M_{n-1} corresponding to eigenvalue $R(z)$ is known.

We state this theorem for $n = 1$ and $M = M_1$, and the analogous relation holds for any $n \geqslant 1$. As before, we define $\phi_1(z) = \phi(z)R(z)$.

Theorem 4.1.

(i) *In the case of proposition 4.1(i),*

$$P_z = \frac{1}{\phi(z)\frac{\mathrm{d}}{\mathrm{d}z}R(z)}(P_0 - (D-z)^{-1}C)P_{0,R(z)}(P_0 - B(D-z)^{-1}). \tag{21}$$

(ii) *In the case of proposition 4.1(ii),*

$$P_z = (P_0 - (D-z)^{-1}C)(\psi_0(z)M_0 - \psi_1(z))^{-1}(P_0 - B(D-z)^{-1}) \tag{22}$$

where $\psi_0(x) = \phi(x)/(z-x)$ and $\psi_1(x) = \phi_1(x)/(z-x)$. This implies, in particular, that there is an one-to-one map $v_0 \mapsto v = v_0 - (D-z)^{-1}Cv_0$ from $\ell^2(V_0)$ onto the eigenspace of M corresponding to z.

(iii) *In the case of proposition 4.1(iii), the poles of $\phi(z)$ and ϕ_1 are simple and so $R(z)$ has a removable singularity at z, $P_zP_{D,z} = P_z$ and $P_0P_z = 0$, which means that the corresponding eigenfunctions of M vanish on V_0.*

Moreover,

$$\mathrm{rank}\, P_{D,z} - \mathrm{rank}\, P_z = \mathrm{rank}(\psi_0(z)M_0 - \psi_1(z)I_0) = \mathrm{rank}\, P_{0,R(z)}$$

where $\psi_0(x) = \phi(x)(z-x)$ and $\psi_1(x) = \phi_1(x)(z-x)$.

In addition, the following relations hold

$$P_z = P_{D,z} + \frac{1}{\psi_0(z)}P_{D,z}C(M_0 - R(z))^{-1}(I_0 - P_{0,R(z)})BP_{D,z} \tag{23}$$

and $P_{D,z}CP_{0,R(z)} = 0$. Note that $I_0 - P_{0,R(z)}$ is the projector from $\ell^2(V_0)$ onto the space, where $(D-z)^{-1}$ is a well defined bounded operator.

(iv) *In the case of proposition 4.1(iv),*

$$P_z = P_{D,z} + \frac{1}{\phi(z)\frac{\mathrm{d}}{\mathrm{d}z}R(z)}(P_0 - (D-z)^{-1}C)P_{0,R(z)}(P_0 - B(D-z)^{-1}) \tag{24}$$

and the projector $P_{D,z}$ is orthogonal to the second term in the right-hand side of this formula. In particular, $P_zP_{D,z} = P_{D,z}$.

(v) *In the case of proposition 4.1(v), P_z is the sum of the right-hand sides in (22) and (24).*

(vi) *In the case of proposition 4.1(vi), provided there are no corresponding eigenfunction at depth n that vanish on V_{n-1}, we have*

$$P_z = \frac{2}{\psi(z)\frac{d^2}{\mathrm{d}z^2}R(z)}(P_0 - (D-z)^{-1}C)P_{0,R(z)}(P_0 - B(D-z)^{-1}). \tag{25}$$

In general, this formula is combined with (23).

(vii) *In the case of proposition 4.1(vii) we formally have $P_z = 0$.*

(viii) *In the case of proposition 4.1(viii) we have $P_z = P_{D,z}$.*

Proof. Item (i) is the same as theorem 3.3; it is inserted here also for the sake of completeness.

To prove item (ii), we pass to the limit as $x \to z$ in formula (12), which can be rewritten as

$$(M-x)^{-1}(D-x)^{-1} + \frac{1}{z-x}(P_0 - (D-x)^{-1}C)(\psi_0(x)M_0 - \psi_1(x))^{-1}(P_0 - B(D-x)^{-1}). \tag{26}$$

Then the statements to be proved follow if we pass to the limit as $x \to z$ in this formula.

To prove item (iii), we again pass to the limit as $x \to z$ in formula (12). We see that $P_0 P_z \neq 0$ if and only if

$$\lim_{x\to z}(x-z)^2(\psi_0(x)M_0 - \psi_1(x)I_0)^{-1} \neq 0,$$

that is only possible if $\frac{\mathrm{d}}{\mathrm{d}z}R(z) = 0$. Therefore $P_0 P_z = 0$ in our case. Relation (23) follows from (12).

Note that

$$\psi_0(z)M_0 - \psi_1(z)I_0 - P_0 M P_{D,z} M P_0$$

if $z \in \sigma(D)$. Hence $\operatorname{rank}(\psi_0(z)M_0 - \psi_1(z)I_0) = \operatorname{rank}(P_{D,z} - P_z)$. In addition, we have that $\psi_0(z)M_0 - \psi_1(z)I_0$ is nonpositive.

Also we see that $P_0(M-z)^{-1}P_0$ is a bounded operator on $\ell^2(V_0)$ and so we have $P_0(M-z)^{-1}P_0 = \lim_{x\to z}(z-x)(\psi_0(x)M_0 - \psi_1(x)I_0)^{-1}$. Hence $P_0(M-z)^{-1}P_0 = 0$ if and only if $R(z)$ has a pole at z or $R(z) \in \rho(M_0)$. If $R(z)$ has a removable singularity at z then

$$\psi_0(z)\frac{\mathrm{d}}{\mathrm{d}z}R(z)P_0(M-z)^{-1}P_0 = P^0_{R(z)}.$$

To prove item (iv), note that the relation $P_z P_{D,z} = P_{D,z}$ easily follows from the fact that ϕ and ϕ_1 do not have poles. Then, if we restrict everything to the orthogonal complement of the image of $P_{D,z}$, we can apply item (i) of this theorem.

Item (v) follows from items (ii) and (iv). The proof of item (vi) is a combination of the proofs of items (i) and (iii). Items (vii) and (viii) easily follow from (12). □

5. Level-3 Sierpiński gasket

The level-3 Sierpiński gasket is shown in figure 1. It has been used as an example in several works [6, 28, 55, and references therein]. In particular, the spectrum was computed in the recent paper [14] independently of our work.

The matrix for the depth-1 Laplacian $M_1 = M$ is

$$M = \begin{pmatrix}
1 & 0 & 0 & -\frac12 & -\frac12 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & -\frac12 & -\frac12 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & -\frac12 & -\frac12 & 0\\
-\frac14 & 0 & 0 & 1 & -\frac14 & 0 & 0 & 0 & -\frac14 & -\frac14\\
-\frac14 & 0 & 0 & -\frac14 & 1 & -\frac14 & 0 & 0 & 0 & -\frac14\\
0 & -\frac14 & 0 & 0 & -\frac14 & 1 & -\frac14 & 0 & 0 & -\frac14\\
0 & -\frac14 & 0 & 0 & 0 & -\frac14 & 1 & -\frac14 & 0 & -\frac14\\
0 & 0 & -\frac14 & 0 & 0 & 0 & -\frac14 & 1 & -\frac14 & -\frac14\\
0 & 0 & -\frac14 & -\frac14 & 0 & 0 & 0 & -\frac14 & 1 & -\frac14\\
0 & 0 & 0 & -\frac16 & -\frac16 & -\frac16 & -\frac16 & -\frac16 & -\frac16 & 1
\end{pmatrix}$$

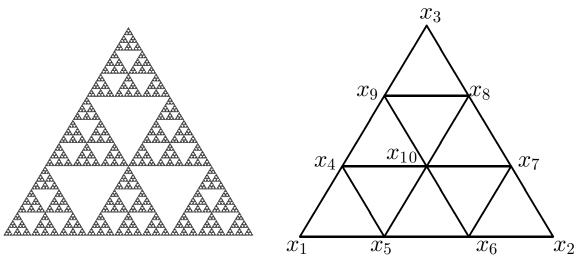

Figure 1. The level-3 Sierpiński gasket and its V_1 network.

and the eigenfunction extension map $(D-z)^{-1}C$ is

$$\begin{pmatrix}
\frac{-24+109z-132z^2+48z^3}{3(1-6z+4z^2)(15-32z+16z^2)} & \frac{-9+7z}{3(1-6z+4z^2)(15-32z+16z^2)} & \frac{-4+3z}{3(-5+34z-4z^2+16z^3)} \\
\frac{-24+109z-132z^2+48z^3}{3(1-6z+4z^2)(15-32z+16z^2)} & \frac{-4+3z}{3(-5+34z-4z^2+16z^3)} & \frac{-9+7z}{3(1-6z+4z^2)(15-32z+16z^2)} \\
\frac{-4+3z}{3(-5+34z-4z^2+16z^3)} & \frac{-24+109z-132z^2+48z^3}{3(1-6z+4z^2)(15-32z+16z^2)} & \frac{-9+7z}{3(1-6z+4z^2)(15-32z+16z^2)} \\
\frac{-9+7z}{3(1-6z+4z^2)(15-32z+16z^2)} & \frac{-24+109z-132z^2+48z^3}{3(1-6z+4z^2)(15-32z+16z^2)} & \frac{-4+3z}{3(-5+34z-4z^2+16z^3)} \\
\frac{-9+7z}{3(1-6z+4z^2)(15-32z+16z^2)} & \frac{-4+3z}{3(-5+34z-4z^2+16z^3)} & \frac{-24+109z-132z^2+48z^3}{3(1-6z+4z^2)(15-32z+16z^2)} \\
\frac{-4+3z}{3(-5+34z-4z^2+16z^3)} & \frac{-9+7z}{3(1-6z+4z^2)(15-32z+16z^2)} & \frac{-24+109z-132z^2+48z^3}{3(1-6z+4z^2)(15-32z+16z^2)} \\
-\frac{1}{3-18z+12z^2} & -\frac{1}{3-18z+12z^2} & -\frac{1}{3-18z+12z^2}
\end{pmatrix}.$$

Moreover, we compute that

$$\phi(z) = \frac{(2z-3)(6z-7)}{3(4z-5)(4z-3)(1-6z+4z^2)}$$

and

$$R(z) = \frac{6z(z-1)(4z-5)(4z-3)}{6z-7}.$$

The eigenvalues of D, written with multiplicities are

$$\sigma(D) = \left\{\tfrac{3}{2}, \tfrac{1}{4}(3+\sqrt{5}), \tfrac{5}{4}, \tfrac{5}{4}, \tfrac{3}{4}, \tfrac{3}{4}, \tfrac{1}{4}(3-\sqrt{5})\right\}.$$

One can also compute

$$\sigma(M) = \left\{\tfrac{3}{2}, \tfrac{3}{2}, \tfrac{3}{2}, \tfrac{3}{2}, \tfrac{1}{4}(3+\sqrt{2}), \tfrac{1}{4}(3+\sqrt{2}), 1, \tfrac{1}{4}(3-\sqrt{2}), \tfrac{1}{4}(3-\sqrt{2}), 0\right\}.$$

We find that $\phi(z) = 0$ has two solutions $\left\{\frac{7}{6}\right\}, \left\{\frac{3}{2}\right\}$. Thus, the exceptional set is

$$E(M_0, M) = \left\{\tfrac{3}{2}, \tfrac{1}{4}(3+\sqrt{5}), \tfrac{5}{4}, \tfrac{3}{4}, \tfrac{1}{4}(3-\sqrt{5}), \tfrac{7}{6}\right\}.$$

To begin the analysis of the exceptional values, note that find the poles of $R(z)$ and see if it is an exceptional value. It is easy to see that $\frac{3}{4}, \frac{5}{4}, \frac{1}{4}(3-\sqrt{5})$ and $\frac{1}{4}(3+\sqrt{5})$ are poles of $\phi(z)$ and so we can use proposition 4.1 (iii) to compute the multiplicities. We obtain

$$\begin{aligned}
&\mathrm{mult}_1\left(\tfrac{3}{4}\right) = 2-3+1 = 0, && \mathrm{mult}_2\left(\tfrac{3}{4}\right) = 12-10+1 = 3,\\
&\mathrm{mult}_1\left(\tfrac{5}{4}\right) = 2-3+1 = 0, && \mathrm{mult}_2\left(\tfrac{5}{4}\right) = 12-10+1 = 3,\\
&\mathrm{mult}_1\left(\tfrac{3\pm\sqrt{5}}{4}\right) = 1-3+2 = 0, && \mathrm{mult}_2\left(\tfrac{3\pm\sqrt{5}}{4}\right) = 6-10+4 = 0.
\end{aligned}$$

Note that $R\left(\frac{3}{4}\right) = R\left(\frac{5}{4}\right) = 0$ and $R\left(\frac{3\pm\sqrt{5}}{4}\right) = \frac{3}{2}$. Also, $\frac{3}{2}$ is not a pole of $\phi(z)$ but $\phi\left(\frac{3}{2}\right) = 0$

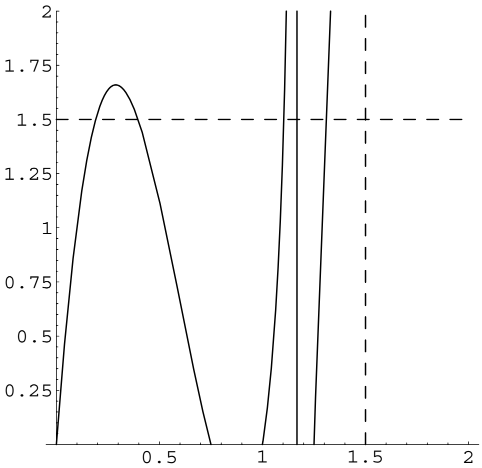

Figure 2. The graph of $R(z)$ for the level-3 Sierpiński gasket.

Table 1. Ancestor-offspring structure of the eigenvalues on the level-3 Sierpiński gasket.

<table>
<tr><td>$z \in \sigma(M_0)$</td><td colspan="10">$\mathbf{0}$</td><td colspan="10">$\frac{3}{2}$</td></tr>
<tr><td>$\mathrm{mult}_0(z)$</td><td colspan="10">1</td><td colspan="10">2</td></tr>
<tr><td>$z \in \sigma(M_1)$</td><td colspan="4">$\mathbf{0}$</td><td colspan="4">$\mathbf{1}$</td><td>$\frac{3}{4}$</td><td>$\frac{5}{4}$</td><td colspan="8">$\frac{3\pm\sqrt{2}}{4}$</td><td colspan="2">$\frac{3\pm\sqrt{5}}{4}$</td><td colspan="4">$\frac{3}{2}$</td></tr>
<tr><td>$\mathrm{mult}_1(z)$</td><td colspan="4">1</td><td colspan="4">1</td><td></td><td></td><td colspan="4">2</td><td colspan="4">2</td><td></td><td></td><td colspan="4">4</td></tr>
<tr><td>$z \in \sigma(M_2)$</td><td>$\mathbf{0}$</td><td>$\mathbf{1}$</td><td>$\frac{3}{4}$</td><td>$\frac{5}{4}$</td><td>*</td><td>*</td><td>*</td><td>*</td><td></td><td></td><td>*</td><td>*</td><td>*</td><td>*</td><td>*</td><td>*</td><td>*</td><td>*</td><td></td><td></td><td colspan="2">$\frac{3\pm\sqrt{2}}{4}$</td><td colspan="2">$\frac{3\pm\sqrt{5}}{4}$</td><td>$\frac{3}{2}$</td><td>$\frac{3}{4}$</td><td>$\frac{4}{5}$</td></tr>
<tr><td>$\mathrm{mult}_2(z)$</td><td>1</td><td>1</td><td></td><td></td><td>1</td><td>1</td><td>1</td><td>1</td><td></td><td></td><td>2</td><td>2</td><td>2</td><td>2</td><td>2</td><td>2</td><td>2</td><td>2</td><td></td><td></td><td>4</td><td>4</td><td></td><td></td><td>16</td><td>3</td><td>3</td></tr>
</table>

and therefore we use proposition 4.1(v) to compute the multiplicities. We obtain

$$\mathrm{mult}_1\left(\tfrac{3}{2}\right) = 1 + 0 + 3 = 4, \qquad \mathrm{mult}_2\left(\tfrac{3}{2}\right) = 6 + 0 + 10 = 16.$$

The ancestor-offspring structure of the eigenvalues on the level-3 Sierpiński gasket is shown in table 1. The multiplicity of the ancestor is the same as that of the offspring by proposition 4.1(i). The empty columns correspond to the exceptional values. If they are eigenvalues of the appropriate M_n, then the multiplicity is shown in the right-hand part of the same row.

Theorem 5.1.

(i) For any $n \geqslant 0$ *we have that* $\sigma(\Delta_n) \subset \bigcup_{m=0}^{n} R_{-m}\left(\left\{0, \frac{3}{2}\right\}\right)$ *and* $\sigma(\Delta_1) = \left\{\frac{3}{2}, \frac{1}{4}(3 \pm \sqrt{2}), 0, 1\right\}$.

(ii) For $n \geqslant 0$ *we have that*

$$\sigma(\Delta_n) = (R_{-n}(0)) \bigcup \left(R_{-(n-1)}\left(\frac{3 \pm \sqrt{5}}{4}\right)\right) \bigcup \left\{\frac{3}{2}\right\}.$$

Table 2. Ancestor-offspring structure of the Dirichlet (zero boundary conditions) eigenvalues on the level-3 Sierpiński gasket.

<table>
<tr><td>$z \in \sigma(M_1^0)$</td><td colspan="4">$\frac{3}{4}$</td><td colspan="4">$\frac{5}{4}$</td><td colspan="8">$\frac{3\pm\sqrt{5}}{4}$</td><td colspan="4">$\frac{3}{2}$</td><td></td><td></td><td></td></tr>
<tr><td>$\mathrm{mult}_1^0(z)$</td><td colspan="4">2</td><td colspan="4">2</td><td colspan="4">1</td><td colspan="4">1</td><td colspan="4">1</td><td></td><td></td><td></td></tr>
<tr><td>$z \in \sigma(M_2^0)$</td><td>*</td><td>*</td><td>*</td><td>*</td><td>*</td><td>*</td><td>*</td><td>*</td><td>*</td><td>*</td><td>*</td><td>*</td><td>*</td><td>*</td><td>*</td><td>*</td><td colspan="2">$\frac{3\pm\sqrt{2}}{4}$</td><td colspan="2">$\frac{3\pm\sqrt{5}}{4}$</td><td>$\frac{3}{2}$</td><td>$\frac{3}{4}$</td><td>$\frac{4}{5}$</td></tr>
<tr><td>$\mathrm{mult}_2^0(z)$</td><td>2</td><td>2</td><td>2</td><td>2</td><td>2</td><td>2</td><td>2</td><td>2</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td></td><td></td><td>13</td><td>5</td><td>5</td></tr>
</table>

(iii) *For* $n \geqslant 0$ *we have* $\dim_n = 3 + \frac{7}{5}(6^n - 1)$.

(iv) *For* $n \geqslant 0$ *we have that* $\mathrm{mult}_n(0) = \mathrm{mult}_n(1) = 1$.

(v) *For* $n \geqslant 2$ *and for* $z \in R_{-k}(1), 0 \leqslant k \leqslant 2$ *we have that* $\mathrm{mult}_n(z) = 1$.

(vi) *For* $n \geqslant 0$ *we have that* $\mathrm{mult}_n\left(\frac{3}{2}\right) = \frac{2\cdot 6^n+8}{5}$.

(vii) *For* $n \geqslant 2$ *and* $0 \leqslant k \leqslant n-2$ *we have for* $z \in R_{-k}\left\{\frac{3}{4}, \frac{5}{4}\right\}$ *that*

$$\mathrm{mult}_n(z) = \tfrac{3}{5}(6^{n-k-1} - 1).$$

Note as a special case $k = 0$ *which gives the multiplicities of* $\frac{3}{4}$ *and* $\frac{5}{4}$.

(viii) *For* $n \geqslant 1$ *with* $0 \leqslant k \leqslant n-1$ *we have that for* $z \in R_{-k}\left(\frac{3\pm\sqrt{2}}{4}\right)$

$$\mathrm{mult}_n(z) = \mathrm{mult}_{n-k-1}\left(\tfrac{3}{2}\right) = \tfrac{1}{5}(2\cdot 6^{n-k-1} + 8).$$

(ix) *For any* $n \geqslant 1$ *with* $0 \leqslant k \leqslant n-1$ *we have that for* $z \in R_{-k}\left(\frac{3\pm\sqrt{5}}{4}\right)$

$$\mathrm{mult}_n(z) = 0.$$

Proof. For this fractal we have $\sigma(\Delta_0) = \left\{0, \frac{3}{2}\right\}$ with $\mathrm{mult}_0\left(\frac{3}{2}\right) = 2$ and, for the purposes of proposition 4.1, $m = 6$.

Item (i) is obtained above in this section.

Item (ii) follows from the subsequent items.

Item (iii) is straightforward by induction.

Item (iv) follows from proposition 4.1(i) because 0 is a fixed point of $R(z)$ and because $R(1) = 0$.

Item (v) easily follows from proposition 4.1(i) and Item (iv).

Item (vi) follows from the previous items and proposition 4.1(v).

Item (vii) follows from proposition 4.1(iii).

Item (viii) follows from proposition 4.1(i).

Item (ix) follows from proposition 4.1(iii), and as a consequence none of these values appear in the spectrum. □

Corollary 5.1. *The normalized limiting distribution of eigenvalues (the integrated density of states) is a pure point measure* κ *with the set of atoms*

$$\left\{\frac{3}{2}\right\} \bigcup \left(\bigcup_{m=0}^{\infty} R_{-m}\left\{\frac{3}{4}, \frac{5}{4}, \frac{3\pm\sqrt{2}}{4}\right\}\right).$$

Moreover, $\kappa\left(\left\{\frac{3}{2}\right\}\right) = \frac{2}{7}$ *and*

$$\kappa(\{z\}) = \tfrac{3}{7}6^{-m-1} \qquad \text{if} \quad z \in R_{-m}\left\{\tfrac{3}{4}, \tfrac{5}{4}\right\};$$

$$\kappa(\{z\}) = \tfrac{2}{7}6^{-m-1} \qquad \text{if} \quad z \in R_{-m}\left\{\tfrac{3\pm\sqrt{2}}{4}\right\}.$$

If one were to consider the situation where the level-3 Sierpiński gasket is bound to fix substrate at its boundary points, the three corners, so that they do not vibrate at all we have only those eigenfunctions left that are part of $\sigma(D)$, that is $\sigma(\Delta_1) = \sigma(D)$. Starting from this spectrum we use the exact same procedure to formulate the multiplicities.

Theorem 5.2. *For this theorem we assume the Dirichlet boundary conditions. (ii) For any* $n \geqslant 1$ *we have that* $\sigma(\Delta_n) \subset \bigcup_{m=0}^{n-1} R_{-m}\left(\left\{\frac{3}{4}, \frac{5}{4}, \frac{3}{2}, \frac{3\pm\sqrt{5}}{4}\right\}\right)$.

(i) For $n \geqslant 1$ *we have* $\dim_n = \frac{7}{5}(6^n - 1)$.

(ii) For $n \geqslant 1$ *we have that* $\mathrm{mult}_n\left(\frac{3}{2}\right) = \frac{12\cdot 6^{n-1}-7}{5}$.

(iii) For $n \geqslant 1$ *and* $0 \leqslant k \leqslant n-1$ *we have for* $z \in R_{-k}\left\{\frac{3}{4}, \frac{5}{4}\right\}$ *that*

$$\mathrm{mult}_n(z) = \tfrac{3}{5}(6^{n-k-1} + 9).$$

Note as a special case $k = 0$ *which gives the multiplicities of* $\frac{3}{4}$ *and* $\frac{5}{4}$.

(iv) For $n \geqslant 2, 0 \leqslant k \leqslant n-2$ *we have that for* $z \in R_{-k}\left(\frac{3\pm\sqrt{2}}{4}\right)$

$$\mathrm{mult}_n(z) = \frac{12\cdot 6^{n-k-2} - 7}{5}.$$

(v) For $n \geqslant 1, 0 \leqslant k \leqslant n-1$ *we have that for* $z \in R_{-k}\left(\frac{3\pm\sqrt{5}}{4}\right)$

$$\mathrm{mult}_n(z) = 0,$$

except in the case $\mathrm{mult}_1\left(\frac{3\pm\sqrt{5}}{4}\right) = 1$.

The proof of this theorem goes in the same way as the proof of theorem 5.1. The ancestor-offspring structure of the Dirichlet eigenvalues on the level-3 Sierpiński gasket is shown in table 2. Note that in the Dirichlet case the normalized limiting distribution of eigenvalues (the integrated density of states) is the same as in corollary 5.1.

6. A fractal 3-tree

The fractal tree is a fractal that is approximated by triangles as shown in figure 3, but in the limit is a topological tree. It appeared as the limit set of the Gupta-Sidki group, see [7, 47, and references therein].

The depth-1 Laplacian matrix $M_1 = M$ is

$$M = \begin{pmatrix} 1 & 0 & 0 & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ -\frac{1}{4} & 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & 0 & 0 \\ 0 & -\frac{1}{4} & 0 & -\frac{1}{4} & 1 & -\frac{1}{4} & 0 & -\frac{1}{4} & 0 \\ 0 & 0 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & 1 & 0 & 0 & -\frac{1}{4} \\ -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & 0 & 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & 0 & 1 \end{pmatrix}$$

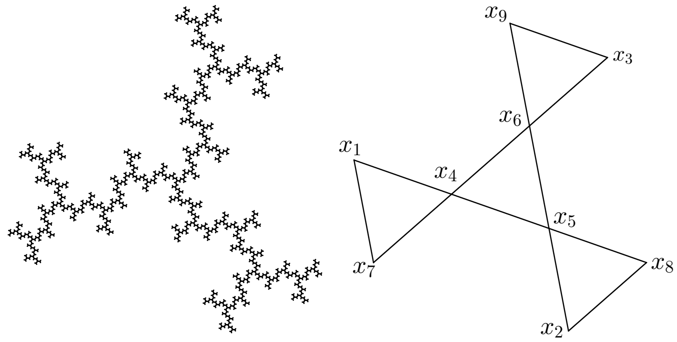

Figure 3. The fractal 3-tree and its V_1 network.

and the eigenfunction extension map $(D-z)^{-1}C$ is

$$\begin{pmatrix} \frac{5+2z(-7+4z)}{9-8z(6+z(-9+4z))} & \frac{2(-1+z)}{(-3+4z)(3+4z(-3+2z))} & \frac{2(-1+z)}{(-3+4z)(3+4z(-3+2z))} \\ \frac{2(-1+z)}{(-3+4z)(3+4z(-3+2z))} & \frac{5+2z(-7+4z)}{9-8z(6+z(-9+4z))} & \frac{2(-1+z)}{(-3+4z)(3+4z(-3+2z))} \\ \frac{2(-1+z)}{(-3+4z)(3+4z(-3+2z))} & \frac{2(-1+z)}{(-3+4z)(3+4z(-3+2z))} & \frac{5+2z(-7+4z)}{9-8z(6+z(-9+4z))} \\ \frac{-7+8(3-2z)z}{(-3+4z)(3+4z(-3+2z))} & \frac{1}{9-8z(6+z(-9+4z))} & \frac{1}{9-8z(6+z(-9+4z))} \\ \frac{1}{9-8z(6+z(-9+4z))} & \frac{-7+8(3-2z)z}{(-3+4z)(3+4z(-3+2z))} & \frac{1}{9-8z(6+z(-9+4z))} \\ \frac{1}{9-8z(6+z(-9+4z))} & \frac{1}{9-8z(6+z(-9+4z))} & \frac{-7+8(3-2z)z}{(-3+4z)(3+4z(-3+2z))} \end{pmatrix}.$$

From here, we compute that

$$\phi(z) = \frac{3-2z}{9-48z+72z^2-32z^3}$$

and

$$R(z) = 4z(z-1)(4z-3).$$

The eigenvalues of D written with multiplicities are

$$\sigma(D) = \left\{\tfrac{3}{2}, \tfrac{3}{2}, \tfrac{1}{4}(3+\sqrt{3}), \tfrac{3}{4}, \tfrac{3}{4}, \tfrac{1}{4}(3-\sqrt{3})\right\}$$

and the corresponding eigenvectors are $\{1,0,-1,-1,0,1\}$, $\{1,-1,0,-1,1,0\}$, $\left\{\frac{1-\sqrt{3}}{2}, \frac{1-\sqrt{3}}{2}, \frac{1-\sqrt{3}}{2}, 1, 1, 1\right\}$, $\left\{-\frac{1}{2}, 0, \frac{1}{2}, -1, 0, 1\right\}$, $\left\{-\frac{1}{2}, \frac{1}{2}, 0, -1, 1, 0\right\}$, and $\left\{\frac{1+\sqrt{3}}{2}, \frac{1+\sqrt{3}}{2}, \frac{1+\sqrt{3}}{2}, 1, 1, 1\right\}$.

Computing the eigenvalues of M with multiplicities gives

$$\sigma(M) = \left\{\tfrac{3}{2}, \tfrac{3}{2}, \tfrac{3}{2}, \tfrac{3}{2}, \tfrac{3}{2}, 1, \tfrac{1}{4}, \tfrac{1}{4}, 0\right\}$$

and the corresponding eigenvectors are $\{0,0,-1,0,0,0,0,0,1\}$, $\{0,-1,0,0,0,0,0,1,0\}$, $\{-1,0,0,0,0,0,1,0,0\}$, $\{1,0,-1,-1,0,1,0,0,0\}$, $\{1,-1,0,-1,1,0,0,0,0\}$, $\{1,1,1,-1,-1,-1,1,1,1\}$, $\left\{-1,0,1,-\frac{1}{2},0,\frac{1}{2},-1,0,1\right\}$, $\left\{-1,1,0,-\frac{1}{2},\frac{1}{2},0,-1,1,0\right\}$, $\{1,1,1,1,1,1,1,1,1\}$.

The only solution of $\phi(z)=0$ is $\frac{3}{2}$. As such, the exceptional set is

$$E(M_0, M) = \left\{\tfrac{3}{2}, \tfrac{3}{4}, \tfrac{1}{4}(3+\sqrt{3}), \tfrac{1}{4}(3-\sqrt{3})\right\}.$$

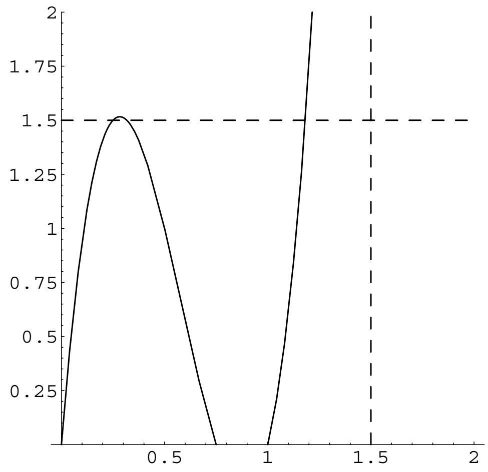

Figure 4. The graph of $R(z)$ for the fractal tree.

Table 3. Ancestor-offspring structure of the eigenvalues of the fractal tree.

<table>
<tr><td>$z \in \sigma(M_0)$</td><td colspan="10">$\mathbf{0}$</td><td colspan="9">$\frac{3}{2}$</td><td colspan="7"></td></tr>
<tr><td>$\text{mult}_0(z)$</td><td colspan="10">1</td><td colspan="9">2</td><td colspan="7"></td></tr>
<tr><td>$z \in \sigma(M_1)$</td><td colspan="4">$\mathbf{0}$</td><td colspan="4">$\mathbf{1}$</td><td>$\frac{3}{4}$</td><td>$\frac{5}{4}$</td><td colspan="8">$\frac{3\pm\sqrt{2}}{4}$</td><td>$\frac{3\pm\sqrt{5}}{4}$</td><td colspan="4">$\frac{3}{2}$</td><td colspan="3"></td></tr>
<tr><td>$\text{mult}_1(z)$</td><td colspan="4">1</td><td colspan="4">1</td><td></td><td></td><td colspan="4">2</td><td colspan="4">2</td><td></td><td colspan="4">4</td><td colspan="3"></td></tr>
<tr><td>$z \in \sigma(M_2)$</td><td>$\mathbf{0}$</td><td>$\mathbf{1}$</td><td>$\frac{3}{4}$</td><td>$\frac{5}{4}$</td><td>*</td><td>*</td><td>*</td><td>*</td><td></td><td></td><td>*</td><td>*</td><td>*</td><td>*</td><td>*</td><td>*</td><td>*</td><td>*</td><td></td><td colspan="2">$\frac{3\pm\sqrt{2}}{4}$</td><td colspan="2">$\frac{3\pm\sqrt{5}}{4}$</td><td>$\frac{3}{2}$</td><td>$\frac{3}{4}$</td><td>$\frac{4}{5}$</td></tr>
<tr><td>$\text{mult}_2(z)$</td><td>1</td><td>1</td><td></td><td></td><td>1</td><td>1</td><td>1</td><td>1</td><td></td><td></td><td>2</td><td>2</td><td>2</td><td>2</td><td>2</td><td>2</td><td>2</td><td>2</td><td></td><td>4</td><td>4</td><td></td><td></td><td>16</td><td>3</td><td>3</td></tr>
</table>

For analysis of exceptional values, one can find $R(z)$ at each exceptional point by

$$R^{-1}(0) = \left\{0, \tfrac{3}{4}, 1\right\} \qquad \text{and} \qquad R^{-1}\left(\tfrac{3}{2}\right) = \left\{\tfrac{1}{4}, \tfrac{1}{4}(3-\sqrt{3}), \tfrac{1}{4}(3+\sqrt{3})\right\}.$$

Using proposition 4.1, one can determine the multiplicities of the exceptional values. For the value $\frac{3}{2}$, which is a zero of $\phi(z)$, we use proposition 4.1(v) to find the multiplicities:

$$\text{mult}_1\left(\tfrac{3}{2}\right) = 4^0(2) + 0 + 3 = 5, \qquad \text{mult}_2\left(\tfrac{3}{2}\right) = 4^1(2) + 0 + 9 = 17.$$

For the value $\frac{3}{4}$, which is a pole of $\phi(z)$, we use proposition 4.1(iii) to find the multiplicities.

$$\text{mult}_1\left(\tfrac{3}{4}\right) = 4^0(2) - 3 + 1 = 0, \qquad \text{mult}_2\left(\tfrac{3}{4}\right) = 4^1(2) - 9 + 1 = 0.$$

For the values $\frac{1}{4}(3+\sqrt{3})$ and $\frac{1}{4}(3-\sqrt{3})$, which are poles of $\phi(z)$, we use proposition 4.1(iii) to find the multiplicities:

$$\text{mult}_1\left(\tfrac{1}{4}(3\pm\sqrt{3})\right) = 4^0(1) - 3 + 2 = 0,$$

$$\text{mult}_2\left(\tfrac{1}{4}(3\pm\sqrt{3})\right) = 4^1(1) - 9 + 5 = 0.$$

The ancestor-offspring structure of the eigenvalues of the Fractal Tree is shown in table 3. As before, the symbol * indicates branches of the inverse function $R^{-1}(z)$ computed at the ancestor value. The multiplicity of the ancestor equals to that of the offspring by proposition 4.1(i). The exceptional values are represented by the empty columns. If they are eigenvalues of the appropriate M_n, then the multiplicity is shown in the right-hand part of the same row.

Theorem 6.1.

(i) For any $n \geqslant 0$ *we have that* $\sigma(\Delta_n) \subset \bigcup_{m=0}^{n} R_{-m}\left(\left\{0, \frac{3}{2}\right\}\right) \bigcup \left\{\frac{3}{2}\right\}$ *and* $\sigma(\Delta_1) = \left\{\frac{3}{2}, \frac{1}{4}(3 \pm \sqrt{3}), \frac{3}{4}\right\}$.

(ii) For $n \geqslant 2$ *we have that*

$$\sigma(\Delta_n) = \left\{0, \frac{3}{2}\right\} \bigcup \left(\bigcup_{k=0}^{n-1} R_{-k} \left\{\frac{1}{4}, 1\right\} \right).$$

And for $n = 1$ *we have* $\sigma(\Delta_1) = \left\{0, \frac{1}{4}, 1, \frac{3}{2}\right\}$.

(iii) For $n \geqslant 0$ *we have* $\dim_n = 3 + 2(4^n - 1)$.

(iv) For $n \geqslant 0$ *we have* $\mathrm{mult}_n(0) = \mathrm{mult}_n(1) = 1$.

(v) For $n \geqslant 2$ *with* $0 \leqslant k \leqslant n-2$ *we have that if* $z \in R_{-k}(1)$ *then*

$$\mathrm{mult}_n(z) = \mathrm{mult}_{n-k}(1) = 1.$$

(vi) For $n \geqslant 0$ *we have that*

$$\mathrm{mult}_n\left(\tfrac{3}{2}\right) = 4^n + 1.$$

(vii) For $n \geqslant 1$ *with* $0 \leqslant k \leqslant n$ *we have for* $z \in R_{-k}\left(\frac{1}{4}\right)$ *that*

$$\mathrm{mult}_n(z) = \mathrm{mult}_{n-k}\left(\tfrac{1}{4}\right) = \mathrm{mult}_{n-k-1}\left(\tfrac{3}{2}\right) = 4^{n-k-1} + 1.$$

(viii) For $n \geqslant 1$ *we have* $\mathrm{mult}_n\left(\frac{3}{4}\right) = 0$.

(ix) For $n \geqslant 1$ *with* $0 \leqslant k \leqslant n$ *we have that if* $z \in R_{-k}\left(\frac{3\pm\sqrt{3}}{4}\right)$ *then* $\mathrm{mult}_n(z) = 0$.

Proof. For this fractal we have $\sigma(\Delta_0) = \left\{0, \frac{3}{2}\right\}$ with $\mathrm{mult}_0\left(\frac{3}{2}\right) = 2$ and, for the purposes of proposition 4.1, $m = 6$.

Item (i) is obtained above in this section.

Item (ii) follows from the subsequent items.

Item (iii) is straightforward by induction.

Item (iv) follows from proposition 4.1(i) because 0 is a fixed point of $R(z)$ and because $R(1) = 0$.

Item (v) easily follows from proposition 4.1(i) and Item (iv).

Item (vi) follows from the previous items and proposition 4.1(v).

Item (vii) follows from proposition 4.1(i).

Items (viii) and (ix) follow from proposition 4.1(iii), and as a consequence none of these values appear in the spectrum. □

Corollary 6.1. *The normalized limiting distribution of eigenvalues (the integrated density of states) is a pure point measure* κ *with the set of atoms*

$$\left\{\frac{3}{2}\right\} \bigcup \left(\bigcup_{m=0}^{\infty} R_{-m} \left\{\frac{1}{4}\right\} \right).$$

Moreover, $\kappa\left(\left\{\frac{3}{2}\right\}\right) = \frac{1}{2}$, *and* $\kappa(\{z\}) = \frac{1}{2}4^{-m-1}$ *if* $z \in R_{-m}\left\{\frac{1}{4}\right\}$.

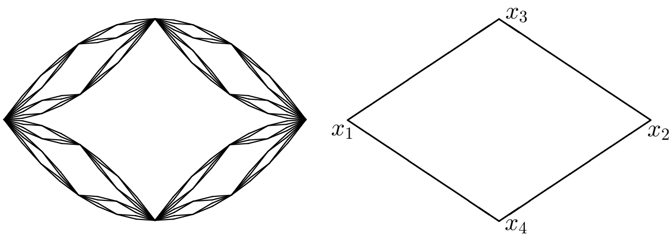

Figure 5. The diamond fractal and its V_1 network.

An analysis of the eigenvalues of the 3-tree with Dirichlet boundary conditions does not appear to have any physical implications as the choice of the three specific boundary points of this fractal would not be a natural choice for where a physical object would be attached to a substrate or further system. It would seem to the authors that a more natural selection of attachment points would be at many more than just the three boundary points and this would very much complicate the analysis.

7. Diamond fractal

The diamond fractal is shown in figure 5. The diamond self-similar hierarchical lattice appeared as an example in several physics works, such as [22]. Recently the critical percolation on the diamond fractal was analyzed in [23].

We can modify the standard results for the unit interval [0,1], see for instance [4], to develop the spectral decimation method for the diamond fractal. The matrix of the depth-1 Laplacian $M_1 = M$ is

$$M = \begin{pmatrix} 1 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 & 1 \end{pmatrix}$$

and the eigenfunction extension map is now the square matrix with the same entries

$$(D - z)^{-1}C = \frac{1}{2(z-1)}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

while the functions

$$\phi(z) = \frac{1}{2(1-z)}$$

and

$$R(z) = 2z(2-z)$$

are the same as for the unit interval, $\sigma(D) = \{1, 1\}$ has multiplicity two, and $\sigma(M) = \{2, 1, 1, 0\}$ with the corresponding eigenvectors $\{-1, -1, 1, 1\}$, $\{-1, 1, 0, 0\}$, $\{0, 0, -1, 1\}$, $\{1, 1, 1, 1\}$. The exceptional set is

$$E(M_0, M) = \{1\}.$$

Theorem 7.1.

(i) For any $n \geqslant 0$ *we have that*

$$\sigma(\Delta_n) = \bigcup_{m=0}^{n} R_{-m}(\{0, 2\}).$$

(ii) For any $n \geqslant 0$ *we have* $\dim_n = 3 + 2(4^n - 1)$.

(iii) For any $n \geqslant 0$ *we have* $\mathrm{mult}_n(0) = \mathrm{mult}_n(2) = 1$.

(iv) For any $n \geqslant 1$ *and* $0 \leqslant k \leqslant n-1$ *we have* $\mathrm{mult}_n(z) = \frac{4^{n-k}+2}{3}$ *if* $z \in R_{-k}(1)$.

Proof. Item (i) follows from (iii) and (iv).

Item (ii) is obtained by induction.

Item (iii) follows from proposition 4.1(i), and the fact that $R(0) = R(2) = 0$.

Item (iv): For the analysis of the only exceptional value $z = 1$, note that it is a pole of $\phi(z)$, $R(1) = 2$, $R(z)$ has a removable singularity at 1, and $\frac{\mathrm{d}}{\mathrm{d}z}R(1) = 0$. Therefore by proposition 4.1(vi) we have

$$\mathrm{mult}_n(1) = 4^{n-1} \cdot 2 - \frac{2 \cdot 4^{n-1} + 4}{3} + 2 = \frac{4^n + 2}{3}$$

for all $n \geqslant 1$. This implies Item (iv). □

Corollary 7.1. *The normalized limiting distribution of eigenvalues (the integrated density of states) is a pure point measure* κ *with the set of atoms*

$$\bigcup_{m=0}^{\infty} R_{-m}\{1\}$$

and $\kappa(\{z\}) = \frac{1}{2}4^{-m}$ *if* $z \in R_{-m}\{1\}$.

8. Conclusions

We considered the classical Laplacian on fractals, which generalizes the usual one-dimensional second derivative, is the generator of the self-similar diffusion process, and has possible applications as the quantum Hamiltonian. We proved that for a large class of self-similar fully symmetric finitely ramified fractals one can compute eigenvalues, eigenfunctions and their multiplicities by matrix analysis, including analysis of singularities. Our analysis, in particular, allowed the computation of the limiting distribution of eigenvalues (i.e., integrated density of states), which is a pure point measure (except the case of the usual one-dimensional interval), and the spectral zeta function of the fractals, which can potentially allow the use of zeta regularization techniques. As examples we considered the level-3 Sierpinski gasket, a fractal 3-tree and the diamond fractal.

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